ON DIRECT PRODUCT SUBGROUPS OF $\mathrm{SO}_3(\mathbb{R})$

DIEGO RATTAGGI

ABSTRACT. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that the two factors G_1 and G_2 are non-trivial groups. We show that if $G_1 \times G_2$ is not abelian, then one factor is the (abelian) group of order 2, and the other factor is non-abelian and contains an element of order 2. There exist finite and infinite such non-abelian subgroups.

Let F_2 be the free group of rank 2. It is well-known that the group $\mathrm{SO}_3(\mathbb{R})$ has subgroups isomorphic to F_2, e.g.

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} -3/5 & 0 & 4/5 \\ 0 & 1 & 0 \\ -4/5 & 0 & -3/5 \end{pmatrix} \right\rangle_{\mathrm{SO}_3(\mathbb{R})} \cong F_2,$$

and subgroups isomorphic to $\mathbb{Z} \times \mathbb{Z}$, like

$$\left\langle \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}, \begin{pmatrix} 1 & 0 & 0 \\ 0 & -15/17 & -8/17 \\ 0 & 8/17 & -15/17 \end{pmatrix} \right\rangle_{\mathrm{SO}_3(\mathbb{R})} \cong \mathbb{Z} \times \mathbb{Z}.$$

However, $\mathrm{SO}_3(\mathbb{R})$ has no subgroups isomorphic to $\mathbb{Z} \times F_2$. More precisely, if $G_1 \times G_2$ is a non-abelian subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_1, G_2 are non-trivial, then G_1, G_2 both contain an element of order 2, and moreover G_1 or G_2 is abelian. We will give an elementary proof of these results (Proposition 7 and Proposition 14) using the Hamilton quaternion algebra $\mathbb{H}(\mathbb{R})$. Additionally, we will show in Proposition 16 that any non-trivial element in the abelian factor has order 2 and in Theorem 18 that in fact the abelian factor is the group of order 2.

Recall that elements $x \in \mathbb{H}(\mathbb{R})$ are of the form $x = x_0 + x_1 i + x_2 j + x_3 k$, where $x_0, x_1, x_2, x_3 \in \mathbb{R}$, and multiplication in $\mathbb{H}(\mathbb{R})$ is induced by the rules $i^2 = j^2 = k^2 = -1$ and $ij = -ji = k$. The *norm* of x is by definition $|x|^2 = x_0^2 + x_1^2 + x_2^2 + x_3^2 \in \mathbb{R}$. We say that $x, y \in \mathbb{H}(\mathbb{R})$ are *perpendicular* (denoted by $x \perp y$), if $x_1 y_1 + x_2 y_2 + x_3 y_3 = 0$ (i.e. if $(x_1, x_2, x_3)^T$, $(y_1, y_2, y_3)^T$ are perpendicular as vectors in $\mathbb{R}^3$). There is a surjective homomorphism ϑ from the multiplicative group $\mathbb{H}(\mathbb{R}) \setminus \{0\}$ to $\mathrm{SO}_3(\mathbb{R})$ defined by

$$x \mapsto \frac{1}{|x|^2} \begin{pmatrix} x_0^2 + x_1^2 - x_2^2 - x_3^2 & 2(x_1 x_2 - x_0 x_3) & 2(x_1 x_3 + x_0 x_2) \\ 2(x_1 x_2 + x_0 x_3) & x_0^2 - x_1^2 + x_2^2 - x_3^2 & 2(x_2 x_3 - x_0 x_1) \\ 2(x_1 x_3 - x_0 x_2) & 2(x_2 x_3 + x_0 x_1) & x_0^2 - x_1^2 - x_2^2 + x_3^2 \end{pmatrix}.$$

It is easy to check that

$$\ker(\vartheta) = Z(\mathbb{H}(\mathbb{R}) \setminus \{0\}) = \{x \in \mathbb{H}(\mathbb{R}) \setminus \{0\} : x = x_0\}$$

Date: February 2, 2008.
Supported by the Swiss National Science Foundation, No. PP002–68627.

which we will identify with $\mathbb{R} \setminus \{0\}$. Note that if $x \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$, then the axis of the rotation $\vartheta(x)$ is the line through $(0,0,0)^T$ and $(x_1, x_2, x_3)^T$ in $\mathbb{R}^3$. Next, we prove three basic lemmas about (anti-)commutation of quaternions.

Lemma 1. *Let $x, y \in \mathbb{H}(\mathbb{R}) \setminus \{0\}$. Then $xy = -yx$, if and only if $x_0 = y_0 = 0$ and $x \perp y$.*

Proof. Only using quaternion multiplication, we get $xy = -yx$ if and only if the following four equations hold:

$$\begin{aligned} x_1y_1 + x_2y_2 + x_3y_3 &= x_0y_0 \\ x_0y_1 + x_1y_0 &= 0 \\ x_0y_2 + x_2y_0 &= 0 \\ x_0y_3 + x_3y_0 &= 0. \end{aligned}$$

Thus if $x_0 = y_0 = 0$ and $x \perp y$, then clearly $xy = -yx$.

To prove the converse, suppose that $xy = -yx$ and (by contradiction) $x_0 \neq 0$. Then from the four equations, we have $x_0y_0 - x_1y_1 - x_2y_2 - x_3y_3 = 0$ and

$$y_1 = \frac{-x_1y_0}{x_0}, \quad y_2 = \frac{-x_2y_0}{x_0}, \quad y_3 = \frac{-x_3y_0}{x_0}.$$

It follows that

$$x_0y_0 + \frac{x_1^2y_0}{x_0} + \frac{x_2^2y_0}{x_0} + \frac{x_3^2y_0}{x_0} = 0$$

and therefore $y_0|x|^2 = 0$. Since $|x|^2 \geq x_0^2 > 0$, we get $y_0 = 0$ which implies $y_1 = 0$, $y_2 = 0$ and $y_3 = 0$, hence the contradiction $y = 0$, and we conclude $x_0 = 0$. The four original equations become $x_1y_1 + x_2y_2 + x_3y_3 = 0$ (i.e. $x \perp y$ as required) and $x_1y_0 = 0$, $x_2y_0 = 0$, $x_3y_0 = 0$, which implies $y_0 = 0$ (using $x \neq 0$) and we are done. $\square$

Lemma 2. *Two quaternions $x, y \in \mathbb{H}(\mathbb{R})$ commute, if and only if $(x_1, x_2, x_3)^T$ and $(y_1, y_2, y_3)^T$ are linearly dependent over $\mathbb{R}$.*

Proof. This follows from the computation

$$\begin{aligned} xy - yx &= 2(x_2y_3 - x_3y_2)i + 2(x_3y_1 - x_1y_3)j + 2(x_1y_2 - x_2y_1)k \\ &= 2\begin{vmatrix} i & x_1 & y_1 \\ j & x_2 & y_2 \\ k & x_3 & y_3 \end{vmatrix}. \end{aligned}$$

$\square$

Lemma 3. *Let $x, y, z \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$. If $xy = yx$ and $xz = zx$, then $yz = zy$. In other words, the group $\mathbb{H}(\mathbb{R}) \setminus \{0\}$ is commutative transitive on non-central elements.*

Proof. By assumption we have

$$\begin{pmatrix} x_1 \\ x_2 \\ x_3 \end{pmatrix}, \begin{pmatrix} y_1 \\ y_2 \\ y_3 \end{pmatrix}, \begin{pmatrix} z_1 \\ z_2 \\ z_3 \end{pmatrix} \neq \begin{pmatrix} 0 \\ 0 \\ 0 \end{pmatrix}.$$

The statement follows now directly from Lemma 2. $\square$

To describe the structure of direct product subgroups of $\mathrm{SO}_3(\mathbb{R})$, we give some general definitions.

Definition 4. We call a direct product $G_1 \times G_2$ *non-trivial*, if both G_1 and G_2 are non-trivial groups.

Definition 5. We say that the group G satisfies property

- (P_1), if G is abelian.
- (P_2), if G is CSA, i.e. if all its maximal abelian subgroups are malnormal (in other words, if for any maximal abelian subgroup $H < G$ and any $g \in G \setminus H$ the intersection of gHg^{-1} with H is trivial).
- (P_3), if G is commutative transitive, i.e. if $xy = yx$, $xz = zx$ always implies $yz = zy$ (provided $x, y, z \in G \setminus \{1\}$).
- (P_4), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian (equivalently, if in any non-trivial direct product subgroup $G_1 \times G_2 < G$ both factors G_1, G_2 are abelian).
- (P_5), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or exactly one factor is the abelian group of order 2 and the other factor is a non-abelian group containing an element of order 2.
- (P_6), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or exactly one factor is abelian such that the non-abelian factor contains an element of order 2 and any non-trivial element in the abelian factor has order 2.
- (P_7), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian or both factors G_1, G_2 contain an element of order 2.
- (P_8), if any torsion-free non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian.
- (P_9), if G contains no subgroup $\mathbb{Z} \times F_2$.
- (P_{10}), if G contains no subgroup $F_2 \times F_2$.
- (R_3), if G is commutative transitive on non-central elements, i.e. if $xy = yx$, $xz = zx$ always implies $yz = zy$ (provided $x, y, z \in G \setminus ZG$).
- (R_4), if any non-trivial direct product subgroup $G_1 \times G_2 < G$ is abelian, or one factor is non-abelian and the other factor is contained in the center of G.
- (R_6), if in any non-trivial direct product subgroup $G_1 \times G_2 < G$ at least one factor is abelian.

Remark 6. The arrows in the following diagram stand for implications. For example "$(P_1) \longrightarrow (P_2)$" means "if a group G satisfies property (P_1), then G satisfies property (P_2)". These implications follow directly from the given definitions, except maybe $(P_2) \longrightarrow (P_3)$ which is also easy to prove, see [1, Proposition 7].

$$\begin{tikzcd}
(P_2) \arrow[r] & (P_3) \arrow[r] \arrow[d] & (P_4) \arrow[r] \arrow[d] & (P_5) \arrow[r] & (P_6) \arrow[r] \arrow[d] & (P_7) \arrow[r] & (P_8) \arrow[d] \\
(P_1) \arrow[u] & (R_3) \arrow[r] & (R_4) \arrow[rr] & & (R_6) \arrow[r] & (P_{10}) & (P_9) \arrow[l]
\end{tikzcd}$$

We will show in Proposition 7 that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (P_7), and in Proposition 14 that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (R_6), using the map ϑ and our lemmas on quaternions. These results will be refined in Proposition 16 and Theorem 18 to prove that $\mathrm{SO}_3(\mathbb{R})$ satisfies property (P_6) and (P_5).

For a group with trivial center, e.g. for $\mathrm{SO}_3(\mathbb{R})$, properties (P_4) and (R_4) are equivalent. In Observation 13, we illustrate by two examples that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_4) (and hence does not satisfy property (R_4)). As a preparation, Observation 11 shows that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_3).

Proposition 7. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_7).

Proof. Let $G_1 \times G_2$ be a non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$ and suppose that G_1 or G_2 does not contain an element of order 2. We have to prove that $G_1 \times G_2$ is abelian. Let E be the identity matrix in $\mathrm{SO}_3(\mathbb{R})$, and take any $A \in G_1 \setminus \{E\}$, $B, C \in G_2 \setminus \{E\}$. Then $AB = BA$ and $AC = CA$. Take any $x, y, z \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$ such that $\vartheta(x) = A$, $\vartheta(y) = B$ and $\vartheta(z) = C$. We have $\vartheta(x)\vartheta(y) = \vartheta(y)\vartheta(x)$, hence $xyx^{-1}y^{-1} \in \ker(\vartheta)$, i.e. $xy = \lambda yx$ for some $\lambda \in \mathbb{R} \setminus \{0\}$. Taking the norm, and using the rule $|xy|^2 = |x|^2|y|^2$, we see that $\lambda \in \{-1, 1\}$, in other words $xy = yx$ or $xy = -yx$. Similarly, $AC = CA$ implies that $xz = zx$ or $xz = -zx$.

In the case $xy = -yx$, we get $x_0 = y_0 = 0$ by Lemma 1. But then $x^2, y^2 \in \mathbb{R} \setminus \{0\}$ and $A^2 = \vartheta(x^2) = E$, $B^2 = \vartheta(y^2) = E$, hence both G_1 and G_2 contain an element of order 2, a contradiction to our assumption. In the same way, if $xz = -zx$, then we get the contradiction $A^2 = C^2 = E$.

Hence we always have $xy = yx$ and $xz = zx$. Using Lemma 3, we get $yz = zy$ and therefore $BC = CB$. This shows that G_2 is abelian. Similarly, taking two matrices in $G_1 \setminus \{E\}$ and one matrix in $G_2 \setminus \{E\}$, one shows that G_1 is abelian. □

Corollary 8. *The group* $\mathrm{SO}_3(\mathbb{R})$ *contains no subgroup* $\mathbb{Z} \times F_2$ *and no subgroup* $F_2 \times F_2$.

Proof. Property (P_7) implies property (P_9) and (P_{10}). □

Remark 9. A group is called *coherent* if every finitely generated subgroup is finitely presented. Any group containing a subgroup $F_2 \times F_2$ is incoherent. Therefore the non-existence of subgroups $F_2 \times F_2$ is a necessary condition for coherence, although it is not a sufficient condition since there are for example incoherent (hyperbolic) groups (using [2]) not containing $\mathbb{Z} \times F_2$ subgroups. It is a question of Serre ([3, p.734]) whether $\mathrm{GL}_3(\mathbb{Q})$ is coherent.

Question 10. *Is* $\mathrm{SO}_3(\mathbb{R})$ *coherent?*

Using the idea of the proof of Proposition 7, we see that any subgroup of $\mathrm{SO}_3(\mathbb{R})$ which does not contain elements of order 2 (in particular any torsion-free subgroup of $\mathrm{SO}_3(\mathbb{R})$) is commutative transitive. However $\mathrm{SO}_3(\mathbb{R})$ itself is not commutative transitive:

Observation 11. *The group* $\mathrm{SO}_3(\mathbb{R})$ *does not satisfy property* (P_3).

This observation will directly follow from Observation 13, but we give a short alternative proof here.

Proof. Take

$$A := \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}, \; B := \begin{pmatrix} 1 & 0 & 0 \\ 0 & 0 & -1 \\ 0 & 1 & 0 \end{pmatrix}, \; C := \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix},$$

then $AB = BA$ and $AC = CA$, but $BC \neq CB$.

Note that $A = \vartheta(i)$, $B = \vartheta(1 + i)$, $C = \vartheta(j)$ and $i(i + 1) = (i + 1)i$, $ij = -ji$, $(i + 1)j \neq \pm j(i + 1)$. □

Corollary 12. *There is a group G which is commutative transitive on non-central elements, but such that $G/Z(G)$ is not commutative transitive on non-central elements (and therefore such that $G/Z(G)$ is not commutative transitive).*

Proof. Take $G = \mathbb{H}(\mathbb{R}) \setminus \{0\}$ such that $G/ZG \cong \mathrm{SO}_3(\mathbb{R})$ and note that $Z(\mathrm{SO}_3(\mathbb{R}))$ is the trivial group. $\square$

The matrices A, B, C from the proof of Observation 11 generate a non-abelian subgroup $\langle A, B, C\rangle$ of $\mathrm{SO}_3(\mathbb{R})$. However, this group cannot be used to prove that $\mathrm{SO}_3(\mathbb{R})$ does not satisfy property (P_4), since $A = B^2$ and $\langle A, B, C\rangle = \langle B, C\rangle$ is the dihedral group of order 8 which is not decomposable as a non-trivial direct product. Nevertheless, there *are* non-abelian non-trivial direct product subgroups of $\mathrm{SO}_3(\mathbb{R})$.

Observation 13. *The group* $\mathrm{SO}_3(\mathbb{R})$ *does not satisfy property* (P_4).

Proof. We give two examples of a non-abelian non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$, at first an infinite example.

Let $A = \vartheta(i)$, $C = \vartheta(j)$ as in the proof of Observation 11 and let

$$\tilde{B} := \vartheta(1 + 2i) = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -3/5 & -4/5 \\ 0 & 4/5 & -3/5 \end{pmatrix}.$$

We claim that $\langle A, \tilde{B}, C\rangle$ is a non-abelian non-trivial direct product subgroup of $\mathrm{SO}_3(\mathbb{R})$.

First we want to show by contradiction that $A \notin \langle \tilde{B}, C\rangle$. Since $C\tilde{B} = \tilde{B}^{-1}C$ and $C\tilde{B}^{-1} = \tilde{B}C$, any word in the letters $\tilde{B}$, $\tilde{B}^{-1}$, $C = C^{-1}$ can be brought to the form $\tilde{B}^n C$ or $\tilde{B}^n$ for some $n \in \mathbb{Z}$. If we suppose that $A \in \langle \tilde{B}, C\rangle$, then, looking at the upper left entry (which is 1 in A and $\tilde{B}$, but -1 in C), we see that A cannot be written as $\tilde{B}^n C$ and therefore $A = \tilde{B}^n$ for some $n \in \mathbb{Z} \setminus \{0\}$. But since A has order 2, we get $\tilde{B}^{2n} = E$, which contradicts the fact that $\tilde{B}$ has infinite order.

Since $\langle A\rangle$ has only two elements and $A \notin \langle \tilde{B}, C\rangle$, we get $\langle A\rangle \cap \langle \tilde{B}, C\rangle = \{E\}$. Moreover, it is easy to check that A commutes with $\tilde{B}$ and with C. Therefore $\langle A, \tilde{B}, C\rangle < \mathrm{SO}_3(\mathbb{R})$ is a direct product of the group $\langle A\rangle$ of order 2 and the (infinite) non-abelian (solvable) group $\langle \tilde{B}, C\rangle$.

As a finite example we can take the dihedral group of order 12, generated for example by the two matrices

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & 1/2 & -\sqrt{3}/2 \\ 0 & \sqrt{3}/2 & 1/2 \end{pmatrix} \text{ and } \begin{pmatrix} -1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

This group is isomorphic to a direct product of the (non-abelian) dihedral group of order 6 (which is isomorphic to the symmetric group S_3) and the group of order 2. $\square$

Proposition 14. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (R_6).

Proof. Suppose by contradiction that $G_1 \times G_2$ is a non-trivial direct subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_1 and G_2 are non-abelian. First take $A, B \in G_1 \setminus \{E\}$ such that $AB \neq BA$ and $C, D \in G_2 \setminus \{E\}$ such that $CD \neq DC$. Now take $x, y, z, w \in \mathbb{H}(\mathbb{R}) \setminus \mathbb{R}$ such that $\vartheta(x) = A$, $\vartheta(y) = B$, $\vartheta(z) = C$, $\vartheta(w) = D$. Then we have $xy \neq \pm yx$, $zw \neq \pm wz$ and (by the same argument as in the proof of Proposition 7) $xz = \pm zx$, $xw = \pm wx$, $yz = \pm zy$, $yw = \pm wy$.

Suppose that $xz = zx$. If $xw = wx$ then we get by Lemma 3 the contradiction $zw = wz$, hence $xw = -wx$. But then by Lemma 1, $w_0 = 0$ and $x \perp w$. Since

$(x_1, x_2, x_3)^T$ and $(z_1, z_2, z_3)^T$ are linearly dependent by Lemma 2, we conclude $z \perp w$. Since $wz \neq -zw$, we have $z_0 \neq 0$ by Lemma 1, hence $yz = zy$ again by Lemma 1, and $xy = yx$ by Lemma 3, a contradiction.

We have shown that $xz = -zx$. Similarly, it follows that $xw = -wx$, $yz = -zy$ and $yw = -wy$. Lemma 1 implies $x \perp z$ and $x \perp w$. Since $zw \neq wz$, z and w are linearly independent by Lemma 2 and span the plane perpendicular to x. We also have $y \perp z$ and $y \perp w$ by Lemma 1, hence x and y are linearly dependent and we get the contradiction $xy = yx$ by Lemma 2. □

Lemma 15. *Let $A \in \mathrm{SO}_3(\mathbb{R})$ be a rotation of order at least 3. Then the centralizer of A in $\mathrm{SO}_3(\mathbb{R})$ consists of all rotations about the axis of A.*

Proof. Without loss of generality, we may assume that A is a rotation of order at least 3 about the x-axis, hence

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\phi & -\sin\phi \\ 0 & \sin\phi & \cos\phi \end{pmatrix},$$

such that $\sin\phi \neq 0$. Suppose that the matrix

$$B = \begin{pmatrix} b_1 & b_2 & b_3 \\ b_4 & b_5 & b_6 \\ b_7 & b_8 & b_9 \end{pmatrix} \in \mathrm{SO}_3(\mathbb{R})$$

commutes with A. Then $AB = BA$ gives the conditions

$$b_9 \sin\phi = b_5 \sin\phi$$
$$-b_8 \sin\phi = b_6 \sin\phi$$

and

$$b_2(1-\cos\phi) = b_3 \sin\phi$$
$$-b_3(1-\cos\phi) = b_2 \sin\phi$$
$$-b_4(1-\cos\phi) = b_7 \sin\phi$$
$$b_7(1-\cos\phi) = b_4 \sin\phi.$$

The first two equations imply $b_5 = b_9$ and $b_6 = -b_8$. The third and fourth equation imply

$$b_2 = \frac{-b_3(1-\cos\phi)}{\sin\phi} \quad \text{and} \quad \frac{-b_3(1-\cos\phi)^2}{\sin\phi} = b_3 \sin\phi,$$

hence

$$-b_3(1-2\cos\phi) = b_3(\sin^2\phi + \cos^2\phi) = b_3.$$

If $b_3 \neq 0$ then $1 - 2\cos\phi = -1$, hence $\cos\phi = 1$ and we get the contradiction $\sin\phi = 0$. Thus $b_3 = 0$ and $b_2 = 0$. Similarly, the fifth and sixth equation lead to $b_4 = b_7 = 0$, hence

$$B = \begin{pmatrix} b_1 & 0 & 0 \\ 0 & b_5 & -b_8 \\ 0 & b_8 & b_5 \end{pmatrix}$$

We exclude the case $b_1 = -1$ computing the determinant of B, and conclude

$$B = \begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\psi & -\sin\psi \\ 0 & \sin\psi & \cos\psi \end{pmatrix}$$

for some ψ. $\square$

Proposition 16. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_6).

Proof. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_2 is non-abelian and G_1 is abelian and non-trivial. Using Proposition 7 and Proposition 14, it remains to prove that any non-trivial element of G_1 has order 2. Therefore suppose that $A \in G_1 \backslash \{E\}$ has order at least 3. Then by Lemma 15, any element in G_2 is a rotation about the axis of A, which contradicts our assumption that G_2 is non-abelian. $\square$

Lemma 17. *The two matrices*

$$\begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & \sin\phi & -\cos\phi \end{pmatrix}, \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\psi & \sin\psi \\ 0 & \sin\psi & -\cos\psi \end{pmatrix} \in \mathrm{SO}_3(\mathbb{R})$$

commute, if and only if

$$\frac{\phi}{2} - \frac{\psi}{2} \in \{k \cdot \frac{\pi}{2} : k \in \mathbb{Z}\}.$$

In particular, these two $180°$*-rotations commute, if and only if their axes (which lie in the* yz*-plane) are identical or perpendicular.*

Proof. Matrix multiplication gives the condition $\sin\phi \cdot \cos\psi = \cos\phi \cdot \sin\psi$, hence

$$0 = \sin\phi \cdot \cos\psi - \cos\phi \cdot \sin\psi = \sin(\phi - \psi)$$

and

$$\phi - \psi \in \{k \cdot \pi : k \in \mathbb{Z}\}.$$

$\square$

Theorem 18. *The group* $\mathrm{SO}_3(\mathbb{R})$ *satisfies property* (P_5).

Proof. Let $G_1 \times G_2$ be a subgroup of $\mathrm{SO}_3(\mathbb{R})$ such that G_2 is non-abelian and G_1 is abelian and non-trivial. Applying Proposition 16, it remains to show that G_1 has order 2. Let $A \in G_1 \setminus \{E\}$. Without loss of generality we may assume that A is a rotation about the x-axis. It has order 2 by Proposition 16, hence

$$A = \begin{pmatrix} 1 & 0 & 0 \\ 0 & -1 & 0 \\ 0 & 0 & -1 \end{pmatrix}.$$

Any element in $G_1 \setminus \{E\}$ has order 2 and commutes with A. An easy computation shows that if an element in $\mathrm{SO}_3(\mathbb{R})$ commutes with A, then it has either the form

$$\begin{pmatrix} 1 & 0 & 0 \\ 0 & \cos\varphi & -\sin\varphi \\ 0 & \sin\varphi & \cos\varphi \end{pmatrix} \text{ or } \begin{pmatrix} -1 & 0 & 0 \\ 0 & \cos\phi & \sin\phi \\ 0 & \sin\phi & -\cos\phi \end{pmatrix},$$

i.e. it is either a rotation about the x-axis, or a rotation about an axis in the yz-plane by an angle of $180°$. The only element of order 2 of the first form is A itself. Hence if $G_1 \setminus \{E, A\}$ is not empty, then it contains only elements of the second form. Since G_1 is abelian, $G_1 \setminus \{E, A\}$ contains by Lemma 17 at most two elements, and G_1 has therefore at most 4 elements. However, we know by Proposition 7 that also G_2 contains an element of order 2 commuting with A, hence G_1 has less than 4 elements. Since $A \in G_1$ has order 2, we conclude that G_1 has exactly 2 elements. $\square$

Remark 19. All statements in this article remain true if we replace $\mathbb{R}$ by $\mathbb{Q}$. The only construction where we have used irrational numbers was in the second part of Observation 13.

References

[1] Myasnikov, Alexei G.; Remeslennikov, Vladimir N., *Exponential groups. II. Extensions of centralizers and tensor completion of CSA-groups*, Internat. J. Algebra Comput. **6**(1996), no. 6, 687–711.

[2] Rips, Eliyahu, *Subgroups of small cancellation groups*, Bull. London Math. Soc. **14**(1982), no. 1, 45–47.

[3] Serre, Jean-Pierre, *Problem section*, Edited by John Cossey. Proceedings of the Second International Conference on the Theory of Groups (Australian Nat. Univ., Canberra, 1973), pp. 733–740. Lecture Notes in Math., Vol. 372, Springer, Berlin, 1974.

E-mail address: `rattaggi@hotmail.com`